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Determining Sets and Best L_1-Approximation

JOSEPH ROSENBLATT

Department of Mathematics, Ohio State University, Columbus, Ohio 43210

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Let B be a finite-dimensional subspace of $L_1[-1, 1]$. A measurable set $E \subset [-1, 1]$ is a *determining set for* B if $F = f \in B$ on the set E makes f a best approximation to F in B. The existence and structure of these sets are studied in general and for particular subspaces, like the polynomials P_n of degree n or less. If $n \geqslant 2$, then any determining set E for P_n has Lebesgue measure $m(E) > 1$, and also for these determining sets $\lim_{m(E)\to 1} \int_E F\,dm = (1/2)\int_{[-1,1]} F\,dm$ for all $F \in L_1[-1, 1]$. This asymptotic uniform distribution of determining sets for these subspaces is also examined through some explicit constructions of these sets.

Let B be a finite-dimensional subspace of $L_1[-1, 1]$, with Lebesgue measure m on $[-1, 1]$. Suppose that an element $F \in L_1[-1, 1]$ and an element $f_0 \in B$ agree on a measurable subset $E \subset [-1, 1]$. In this case, what conditions on E guarantee that f_0 is always a *best L_1-approximation* to F in B, i.e., that with $\|f\|_1 = \int_{-1}^{1} |f(s)|\,ds$, we have $\|F - f_0\|_1 \leqslant \|F - f\|_1$ for all $f \in B$? We consider here the existence and structure of such sets. Even in the simpler cases, like that of B consisting of all polynomials of degree 2 or less, these sets exhibit many surprising properties.

A measurable subset $E \subset [-1, 1]$ will be called a *determining set for* B, or just a *determining set*, if whenever $F \in L_1[-1, 1]$ and $f \in B$ agree a.e. on E, then f is a best L_1-approximation to F in B. We will denote the characteristic function of a set $E \subset [-1, 1]$ by 1_E.

PROPOSITION 1. *The measurable set $E \subset [-1, 1]$ is a determining set for B if and only if $2\,\|1_E f\|_1 \geqslant \|f\|_1$ for all $f \in B$.*

Proof. Let $F \in L_1[-1, 1]$ and let $f_0 \in B$ equal F a.e. on E. Assume that E satisfies the integral inequality above. Then for any $f \in B$, $\|1_E(f_0 - f)\|_1 \geqslant \|1_{E^c}(f_0 - f)\|_1$. Hence, we have

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$$\begin{aligned}\|F-f\|_1 &= \|1_E(f_0-f)\|_1 + \|1_{E^c}(F-f)\|_1\\ &\geqslant \|1_E(f_0-f)\|_1 + (\|1_{E^c}(F-f_0)\|_1 - \|1_{E^c}(f_0-f)\|_1)\\ &= \|1_E(f_0-f)\|_1 + \|F-f_0\|_1 - \|1_{E^c}(f_0-f)\|_1\\ &\geqslant \|F-f_0\|_1.\end{aligned}$$

Conversely, suppose $2\|1_E f\|_1 < \|f\|_1$ for some $f \in B$. Let $F = f1_{E^c}$. Now, although $F=0$ on E, we have $\|F-0\|_1 = \|1_{E^c}f\|_1 > \|1_E f\|_1 = \|F-f\|_1$. Hence, E is not a determining set for B. ▮

Remark. The same simple argument can be used to characterize when a determining set gives a unique best L_1-approximation; i.e., whenever $F \in L_1[-1, 1]$ agrees with $f_0 \in B$ on E, then $\|F-f_0\| < \|F-f\|_1$ for all $f \in B$ with $f \neq f_0$. Indeed, E has this property if and only if $2\|1_E f\|_1 > \|f\|_1$ whenever $0 \neq f \in B$. We will say that E is a *unique determining set for B* in this case. Since B is finite-dimensional, every unique determining set E for B contains a smaller set E_0 which is just a determining set for B.

EXAMPLES. (a) Let $E = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$ and let B consist of all polynomials of degree 1 or less. One can check easily that E satisfies the condition in Proposition 1 for B. Indeed, it is enough to verify the condition if $f(x) = x + b$, where $b \geqslant 0$. If $b \geqslant 1$, then $\|f\|_1 = 2b = 2\|1_E f\|_1$. If $1 \geqslant b \geqslant \frac{1}{2}$, then $\|f\|_1 = 1 + b^2$ and $\|1_E f\|_1 = b^2 - b + 1$. So in this case, $2\|f\|_1 \geqslant \|1_E f\|_1$. And if $\frac{1}{2} \geqslant b \geqslant 0$, then $\|f\|_1 = 1 + b^2 \leqslant \frac{5}{4} < \frac{6}{4} = 2\|1_E f\|_1$. Hence, E is a determining set for the linear functions.

(b) An even simpler special case of this proposition says that if $F \in L_1[-1, 1]$ and $F = 0$ on a set of measure at least 1, then $\|F\|_1 \leqslant \|F - c\|_1$ for all constants c.

We will later construct determining sets for subspaces B consisting of continuous functions by a simple explicit method. However, it is worth observing that determining (or uniquely determining) sets always exist.

PROPOSITION 2. *Let $B \subset L_1[-1, 1]$ be a finite-dimensional subspace and let $0 < \delta < 1$. Then there exists a determining set E for B with $m(E) = 1 + \delta$.*

Proof. Since B is finite-dimensional, there exists a δ-net $\{|f_1|, \ldots, |f_n|\}$ for $\{|f| : f \in B, \|f\|_1 = 2\}$. By the Liapounov theorem, there exists a measurable set $E \subset [-1, 1]$ such that $m(E) = 1 + \delta$ and, for all $i = 1, \ldots, n$, $\|1_E f_i\|_1 = ((1+\delta)/2)\|f_i\|_1$. Then for all $f \in B$ with $\|f\|_1 = 2$, there exists some $i = 1, \ldots, n$, with $\|\,|f| - |f_i|\,\| \leqslant \delta$. So

$$\begin{aligned}\| |1_E f| \|_1 &= \| 1_E f_i \|_1 + \int_E (|f| - |f_i|)\, dm \\ &\geqslant ((1+\delta)/2) \|f_i\|_1 - \delta \\ &= 1 + \delta - \delta = (1/2) \|f\|_1 .\end{aligned}$$

By multiplying by a suitable scalar, we have $\| 1_E f\|_1 \geqslant (1/2) \|f\|_1$ for all $f \in B$. By Proposition 1, E is a determining set for B with $m(E) = 1 + \delta$. ∎

Remarks. (a) If the constant 1 happens to be in B, then we would always have $m(E) \geqslant 1$ if E is a determining set for B. Also, Proposition 2 is only worthwhile if δ is small because it is easy to show that as $m(E)$ increases to 2, then in all cases $\inf\{ \| 1_E f\|_1 / \|f\|_1 : 0 \neq f \in B\}$ converges to 1 because B is finite-dimensional. Hence, if $m(E)$ is close enough to 2, depending on B, the set E will always be a determining set for B.

(b) The determining sets shown to exist above can always be assumed to be a finite union of intervals. To do this, first choose any $\delta/2$-net for $\{|f| : f \in B, \|f\|_1 = 2\}$. Then choose the set E as before for this $\delta/2$-net and let S be a finite union of intervals with $m(E \Delta S)$ sufficiently small to guarantee $\| 1_S f\|_1 \geqslant 1 + (\delta/2)$ for all f in the $\delta/2$-net. It can then be shown, as before, that $\| 1_S f\|_1 \geqslant (1/2) \|f\|_1$ for all $f \in B$. Finally, since $m(S)$ can be made arbitrarily close to 1 and since one can always enlarge S by another interval, $m(S)$ can be chosen to be any number in $(1, 2]$.

(c) It is not clear from these arguments when one can have $m(E) = 1$ in Proposition 2. This is not always the case, as we will see later.

It would be worthwhile to have simple determining sets, as in Example (a), where $m(E) = 1$ too, but for subspaces containing higher degree polynomials. However, this is not possible in general. The reason for this lies in the close connection between determining sets and another type of set arising in L_1-approximation.

DEFINITION. A set $E \subset [-1, 1]$ is a *mean set for* B if $2\int_E f\, dm = \int_{[-1,1]} f\, dm$ for all $f \in B$. That is, E is a mean set if the function $b = 1_E - 1_{E^c}$ is in the annihilator $B^\perp \subset L_\infty[-1, 1]$ of B.

The next proposition is in [1] and was based on [3, 4]. This proof is somewhat different and gives the property of mean sets that we will need.

PROPOSITION 3. *Let $F \in L_1[-1, 1]$ and let B be a finite-dimensional subspace of $L_1[-1, 1]$. The following conditions are equivalent*:

(1) $\|F\|_1 \leqslant \|F - f\|_1$ *for all* $f \in B$.

(2) *There exists a mean set E for B such that $F \geqslant 0$ on E and $F \leqslant 0$ on E^c.*

Proof. Clearly (1) is equivalent to having $\|F\|_1 = \|F+B\|_1$, the norm of $F+B$ in the quotient $L_1[-1,1]/B$. Because of the usual isometric identification of $B^\perp$ with the dual of $L_1[-1,1]/B$, (1) occurs if and only if $\|F\|_1 = \sup\{|\int_{[-1,1]} Fh| :: \|h\|_\infty \leqslant 1, h \in B^\perp\}$. Now because $\{h \in B^\perp : \|h\|_\infty \leqslant 1\}$ is a compact convex subset in the $w^* = w(L_\infty, L_1)$ topology, the Krein–Milman theorem says that (1) implies there is an extreme point of $\{h \in B^\perp : \|h\|_\infty \leqslant 1\}$ in the set $\{h \in B^\perp : \|h\|_\infty \leqslant 1$ and $|\int_{[-1,1]} Fh\, dm| = \|F\|_1\}$. Hence, (1) occurs if and only if $\|F\|_1 = |\int_{[-1,1]} Fh\, dm|$ for some extreme point h of the unit ball of $B^\perp$.

We claim that these extreme points all have $|h| = 1$ a.e. Certainly, such a point is an extreme point because it is an extreme point of the unit ball of $L_\infty[-1,1]$ in this case. Conversely, if $|h| < 1$ on a set of positive measure, there exists a measurable set $D \subset [-1,1]$, $m(D) > 0$, and some δ, $0 < \delta < 1$, such that $-1+\delta \leqslant h \leqslant 1-\delta$ on D. By using the Liapounov theorem on a finite linear basis of B, there exists a measurable decomposition $D = D_1 \cup D_2$, with $D_1 \cap D_2 = \varnothing$, and with $\int_{D_i} f\, dm = (1/2) \int_D f\, dm$ for $i = 1, 2$ and any $f \in B$. Now define h_1 and h_2 by $h_1 = (h+\delta)\, 1_{D_1} + (h-\delta) 1_{D_2} + h 1_{D^c}$ and $h_2 = (h-\delta)\, 1_{D_1} + (h+\delta)\, 1_{D_2} + h 1_{D^c}$. For both $i = 1, 2$, $\|h_i\|_\infty \leqslant 1$ and $h_i \in B^\perp$. Also, $h_i \neq h$ for $i = 1, 2$. But also, $2h = h_1 + h_2$; hence, h cannot be an extreme point of the ball of $B^\perp$.

Finally, we know that (1) is equivalent to there being an $h \in B^\perp$ with $|h| = 1$ a.e. and $|\int_{[-1,1]} Fh\, dm| = \|F\|_1$. It is clear that both $E = \{h = 1\}$ and $E^c = \{h = -1\}$ are mean sets for B because $h \in B^\perp$. Also, either $F \geqslant 0$ on E and $F \leqslant 0$ on E^c or vice versa since $\|F\|_1 = |\int_{[-1,1]} Fh\, dm|$. Hence, (1) is equivalent to (2). ∎

EXAMPLES. (a) Suppose the set E has the form $(-\beta, -\alpha) \cup (\alpha, \beta)$, where $0 < \alpha < \beta < 1$. Then E is a mean set for polynomials of degree 3 or less if and only if $1/2 = \beta - \alpha = \beta^3 - \alpha^3$. One can explicitly solve these equations to get $\alpha = (\sqrt{5} - 1)/4$ and $\beta = (\sqrt{5} + 1)/4$. The proposition says that if $F \in L_1[-1,1]$ and q is a polynomial of degree 3 or less with $F \geqslant q$ on E and $F \leqslant q$ on E^c, then q is a best L_1-approximation to F among the cubic polynomials.

(b) It would be interesting to have explicit mean sets for the space P_n of polynomials of degree n or less. This seems to be a fairly difficult algebraic problem. For instance, fix $n \geqslant 1$ and let E have the form $E = F \cup -F$, where $F = (\alpha_1, \alpha_2) \cup (\alpha_3, \alpha_4) \cup \cdots \cup (\alpha_n, 1)$ if n is odd, and $F = (\alpha_1, \alpha_2) \cup (\alpha_3, \alpha_4) \cup \cdots \cup (\alpha_{n-1}, \alpha_n)$ if n is even, for some $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$. Then E is already a mean set for odd degree polynomials by symmetry. A necessary and sufficient condition for E to be a mean set for P_{2n-1} is that the $\{\alpha_i : i = 1, \dots, n\}$ satisfy the system of diophantine equations $(-1)^{n+1}(1/2) = \alpha_1^k - \alpha_2^k + \alpha_3^k - \cdots \pm \alpha_n^k$ for all odd $k \in \{1, \dots, 2n-1\}$. This reduces to (a) if $n = 2$. We believe that there is a unique solution with

$0 < \alpha_1 < \cdots < \alpha_n < 1$ to these equations for all n, but it is difficult to prove this explicitly. One approach that works at least for small n, and may work in general, is to use these equations as Newton identities and find equations determining the coefficients of a polynomial $p(x)$ of degree n with the $\{\alpha_i : i = 1,..., n\}$ as its roots. Then the $\{\alpha_i : i = 1,..., n\}$ can be computed, at least approximately, by Newton's algorithm. This has only been carried out in a few cases and leaves the existence of a mean set E of this form for P_{2n-1} unsolved. I would like to thank Professor D. Shapiro for suggesting this approach.

(c) Suppose $F \in L_1[-1, 1]$ and $F \geqslant 0$. Then Proposition 3 says that there exists a constant C so that $\|F\|_1 > \|F - C\|_1$ if and only if $m\{F = 0\} < 1$.

The next proposition is fundamental for the structure of determining sets.

PROPOSITION 4. *Let B be a finite-dimensional subspace of $L_1[-1, 1]$. If E is a determining set for B, then there exists a mean set E_0 for B with $E_0 \subset E$.*

Proof. Suppose E is a determining set for B. Then whenever $F \in L_1[-1, 1]$ is 0 on E, $\|F\|_1 \leqslant \|F - f\|_1$ for all $f \in B$. Taking $F = 1_{E^c}$ and applying Proposition 3, there exists a mean set E_1 for B with $F \geqslant 0$ on E_1 and $F \leqslant 0$ on E_1^c. But then $E^c \cap E_1^c = \varnothing$ and so $E^c \subset E_1$. That is, E contains the mean set $E_0 = E_1^c$. ∎

This last proposition enables us to give a simple proof that in many cases the subspace B can only have determining sets E with $m(E) > 1$. The condition on B that is needed is the following.

DEFINITION. A subspace B of $L_1[-1, 1]$ is said to be *positive-part dense* if the constant $1 \in B$ and if the cone generated by the positive parts $\{f^+ : f \in B\}$ is L_1-norm dense in $\{F \in L_1[-1, 1] : F \geqslant 0\}$.

This condition is not really very restrictive. Since our main interest is in determining sets for spaces of polynomials, we will just prove this proposition by way of providing examples.

PROPOSITION 5. *Let B be a finite-dimensional subspace of $L_1[-1, 1]$ which contains all the quadratic polynomials and also the constant* 1. *Then B is positive-part dense.*

Proof. By the separation theorem, we need only show that if $h \in L_\infty[-1, 1]$ with $\int_{[-1,1]} hq^+ \, dm \geqslant 0$ for all quadratic polynomials, then $h \geqslant 0$. Suppose this implication fails for h. Then there exists $\delta > 0$ such that $L = \{h \leqslant -\delta\}$ has $m(L) > 0$. Now for any ε, $0 < \varepsilon < 1$, there exists an interval $I = (\alpha, \beta) \subset [-1, 1]$ such that $m(L \cap I) > (1 - \varepsilon)\, m(I)$. Let $q(x) =$

$-(4/(\beta-\alpha)^2)(x-\alpha)(x-\beta)$. Then $q^+ = 1_{(\alpha,\beta)}q$. Also, $\max_{[-1,1]} q^+ = q((\alpha+\beta)/2) = 1$. We also observe that $q((1/3)\alpha + (2/3)\beta) = q((2/3)\alpha + (1/3)\beta) = 8/9$. So

$$0 \leqslant \int_{[-1,1]} hq^+ \, dm = \int_I hq^+ \, dm$$

$$= \int_{I\cap L^c} hq^+ \, dm + \int_{I\cap L} hq^+ \, dm$$

$$\leqslant \|h\|_\infty \, m(I\cap L^c) + \int_{I\cap L} hq^+ \, dm$$

because $0 \leqslant q^+ \leqslant 1$ on $[-1, 1]$. Hence,

$$0 \leqslant \|h\|_\infty \, (\varepsilon m(I)) - \delta \int_{I\cap L} q^+ \, dm.$$

If J is the interval from $(2/3)\alpha + (1/3)\beta$ to $(1/3)\alpha + (2/3)\beta$, then we have

$$\begin{aligned} 0 &\leqslant \varepsilon \|h\|_\infty (\beta-\alpha) - \delta \int_{J\cap L} q^+ \, dm \\ &\leqslant \varepsilon \|h\|_\infty (\beta-\alpha) - (8/9)\, \delta m(J\cap L) \\ &= \varepsilon \|h\| (\beta-\alpha) - (8/9)\, \delta m(I\cap L) + (8/9)\, \delta m((I\backslash J)\cap L) \\ &\leqslant \varepsilon \|h\|_\infty (\beta-\alpha) - (8/9)\, \delta(1-\varepsilon)(\beta-\alpha) \\ &\quad + (8/9)\, \delta(2/3)(\beta-\alpha). \end{aligned}$$

Dividing by $\beta-\alpha$ and then letting $\varepsilon \to 0$ gives $0 \leqslant -(8/9)\delta + (16/27)\delta$, which is impossible because $\delta > 0$. ∎

This proposition shows that $B = P_n$ with $n \geqslant 2$ is a subspace in which the next few propositions on determining sets apply.

PROPOSITION 6. *Suppose B is a subspace of $L_1[-1, 1]$ which is positive-part dense. Then no set E is simultaneously a determining set and a mean set for B.*

Proof. Let E be both a determining set and a mean set for B. Let $h = 1_E - 1_{E^c}$. Then, because $2\|1_E f\| \geqslant \|f\|_1$ for all $f \in B$, we have $\|1_E f\|_1 \geqslant \|1_{E^c} f\|_1$ or $0 \leqslant \int_{[-1,1]} h(f^+ + f^-)\, dm$ for all $f \in B$. But $\int_{[-1,1]} hf\, dm = 0$ for all $f \in B$ and, therefore, $0 \leqslant \int_{[-1,1]} h(f^+ + f^-)\, dm = 2\int_{[-1,1]} hf^+\, dm$ for all $f \in B$. But then because the cone generated by $\{f^+ : f \in B\}$ is dense in $L_1^+[-1, 1]$, this means $0 \leqslant \int_{[-1,1]} hF\, dm$ for all $F \in L_1[-1, 1]$, $F \geqslant 0$. Hence $h \geqslant 0$ and $m(E^c) = 0$. This is impossible because $1 \in B$ and $0 = \int_{[-1,1]} h1\, dm = m(E) - m(E^c)$. ∎

Remark. If $B \subset \{f \in L_1[-1, 1] : \int_{[-1,1]} f\,dm = 0\}$, then $[-1, 1]$ is both a determining set and a mean set for B. In the previous proof, we used the assumption that $1 \in B$ to avoid this, but knowing that $\int_{[-1,1]} f\,dm \neq 0$ for some $f \in B$ would do as well for that part of the proof.

COROLLARY 7. *Suppose B is a finite-dimensional subspace of $L_1[-1, 1]$ which is positive-part dense. Then every determining set E has $m(E) > 1$.*

Proof. By Proposition 4, E contains a mean set E_0 for B. Because $1 \in B$, $m(E_0) = m(E_0^c) = 1$. But then $m(E) \geqslant 1$ and, by Proposition 6, $m(E) \neq 1$ because this would make $E = E_0$ a.e. and force E to be a mean set for B too. ∎

Let us suppose then that B is a finite-dimensional subspace of $L_1[-1, 1]$, like P_n with $n \geqslant 2$. The last proposition shows that one cannot have a determining set with measure 1 as in the case of P_1. At one extreme, if $m(E)$ is close to 2, it will be a determining set for B. But in the other extreme, with $m(E)$ close to 1, the set will have some more specific distribution. If possible, one would like to have determining sets E with $m(E)$ close to 1 which are easily described for computational purposes. However, Corollary 7 already shows that as $m(E) \to 1$, the number of intervals in a simple determining set for positive-part dense subspaces B must tend to infinity. Much more is true in this extreme.

PROPOSITION 8. *Let B be a finite-dimensional subspace of $L_1[-1, 1]$ which is positive-part dense. Let (E_n) be a sequence of determining sets for B with $\lim_{n\to\infty} m(E_n) = 1$. Then for all $F \in L_1[-1, 1]$, $\lim_{n\to\infty} \int_{E_n} F\,dm = (1/2) \int_{[-1,1]} F\,dm$.*

Proof. By Corollary 7 and Proposition 4, we know that each $E_n \supset F_n$ with F_n a mean set for B, and $m(E_n) > 1 = m(F_n)$ for all $n \geqslant 1$. Because $\lim_{n\to\infty} m(E_n \backslash F_n) = 0$, the condition $\| 1_{E_n} f \|_1 \geqslant (1/2) \|f\|_1$ for all $f \in B$ and the assumption that B is finite-dimensional guarantee that there is a sequence (ε_n), $\varepsilon_n > 0$, and $\lim_{n\to\infty} \varepsilon_n = 0$ such that $\| 1_{F_n} f \|_1 \geqslant ((1/2) - \varepsilon_n) \|f\|_1$ for all $f \in B$. Since $\int_{F_n} f\,dm = (1/2) \int_{[-1,1]} f\,dm$ for all $f \in B$, we have this inequality for all $f \in B$:

$$\begin{aligned}
\int_{F_n} f^+\,dm &= (1/2) \int_{F_n} (|f| + f)\,dm \\
&\geqslant (1/2) \left[(1/2) \|f\|_1 + \int_{F_n} f\,dm \right] - (\varepsilon_n/2) \|f\|_1 \\
&= (1/2)((1/2)) \left[\|f\|_1 + \int_{[-1,1]} f\,dm \right] - (\varepsilon_n/2) \|f\|_1 \\
&= (1/2) \int_{[-1,1]} f^+\,dm - (\varepsilon_n/2) \|f\|_1 .
\end{aligned}$$

Now we show $1_{E_n} \to 1/2$ as $n \to \infty$ in the $w^* = w(L_\infty, L_1)$ topology. Because this is a metric topology, it suffices to show that any subsequence of (1_{E_n}) has a further subsequence with this property. By reindexing the subsequence, we may call it (1_{E_n}) again. Now by compactness of the w^* topology, there is a subsequence $(1_{E_{n_i}})$ which converges w^* to some $h \in L_\infty[-1, 1]$ with $0 \leqslant h \leqslant 1$ a.e. $[m]$. Because $\lim_{n\to\infty} m(E_{n_i}) = 1$, $\int_{[-1,1]} h\, dm = 1$. Suppose $h < 1/2$ on a set of positive measure. Then there would be some $f \in B$ with $\int_{[-1,1]} hf^+ \, dm < (1/2) \int_{[-1,1]} f^+ \, dm$ because B is positive-part dense. But then

$$\int_{[-1,1]} hf^+ \, dm = \lim_{i\to\infty} \int_{E_{n_i}} f^+ \, dm = \lim_{i\to\infty} \int_{F_{n_i}} f^+ \, dm$$

because $\lim_{i\to\infty} m(E_{n_i} \backslash F_{n_i}) = 0$. So,

$$\begin{aligned}\int_{[-1,1]} hf^+ \, dm &= \lim_{i\to\infty} \int_{F_{n_i}} f^+ \, dm \\ &\geqslant \lim_{i\to\infty} \left((1/2) \int_{[-1,1]} f^+ \, dm - (\varepsilon_{n_i}/2) \|f\|_1 \right) \\ &= (1/2) \int_{[-1,1]} f^+ \, dm > \int_{[-1,1]} hf^+ \, dm.\end{aligned}$$

This contradiction proves $h \geqslant 1/2$ a.e. But $\int_{[-1,1]} h \, dm = 1$ then also proves that h cannot be larger than $1/2$ on a set of positive measure. That is, $h = 1/2$ a.e. ∎

We see from these propositions that if (E_n) is a sequence of determining sets for P_m, $m \geqslant 2$, and $\lim_{n\to\infty} m(E_n) = 1$, then the sets (E_n) approach uniform distribution on $[-1, 1]$ in the limit. So, for n large, or $m(E_n)$ close to 1, a determining set for these subspaces of $L_1[-1, 1]$ will consist of many intervals (if it is a simple set at all) and will not be as easy to describe as the example for P_1. Explicitly, we have this obvious corollary of Proposition 8.

COROLLARY 9. *Let B be a finite-dimensional subspace of $L_1[-1, 1]$ which is positive-part dense. Let (E_n) be a sequence of determining sets for B with $\lim_{n\to\infty} m(E_n) = 1$. Then for every measurable set $F \subset [-1, 1]$, neither the sequence $(E_n \cap F)$ nor $(E_n^c \cap F)$ converges in measure unless $m(F) = 0$. Also, if $I \subset [-1, 1]$ is a non-degenerate interval, then the total variations $(\operatorname{var}(1_{E_n \cap I}))$ and $(\operatorname{var}(1_{E_n^c \cap I}))$ both tend to infinity as $n \to \infty$.*

Proposition 8 suggests a way of constructing determining sets for subspaces of continuous functions which we will now elaborate. However, Corollary 9 also shows that, for the subspaces P_n, $n \geqslant 2$, in particular, the determining set must become more and more disintegrated as its measure approaches 1. So in carrying out our construction, we would like to know how many intervals will actually be needed to comprise a determining set in terms of how small $m(E) - 1$ has become. This explains the basic direction of the following explicit construction of determining sets.

First, we will need the value of a certain constant. Define the modulus of continuity $w_\delta(f)$ as usual, $w_\delta(f) = \sup\{|f(x) - f(y)| : x, y \in [-1, 1], |x - y| \leqslant \delta\}$. The modulus of continuity is a pseudo norm on P_n and a norm on P_n modulo constants. For a function f which is continuously differentiable on $[-1, 1]$, we can estimate the value of $w_\delta(f)$ by $w_\delta(f) \leqslant \delta \max\{|f'(x)| : x \in [-1, 1]\}$. If f is a polynomial in particular, $f(x) = \sum_{i=1}^n a_i x^i$ for some real numbers $a_1, \ldots, a_n$, then $|f'(x)| \leqslant (n(n+1)/2) \max(|a_1, \ldots, |a_n|)$. Hence, in this case, $w_\delta(f) \leqslant C_n \delta \max(|a_1|, \ldots, |a_n|)$, where $C_n = n(n+1)/2$. Moreover, because P_n is finite-dimensional, there is a smallest constant K_n such that $\max(|a_0|, \ldots, |a_n|) \leqslant K_n \|f\|_1$ for all $f \in P_n$, $f(x) = \sum_{i=0}^n a_i x^i$. We also want to estimate K_n. This seems to be fairly difficult to do exactly, even for the quadratic polynomials. In order to get some estimate for K_n, we use the following lemma, a well-known consequence of properties of the Cebyšev polynomials of the second kind.

LEMMA 10. *The value of the infimum*

$$\inf \left\{ \int_{-1}^{1} |a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n| \, dx : a_0, \ldots, a_{n-1} \text{ are real} \right\}$$

is $1/2^{n-1}$.

This lemma gives us

PROPOSITION 11. *The constants* (K_n) *above satisfy the recurrence* $K_{n+1} \leqslant ((2^{n+1}/n+1) + 1) K_n$.

Proof. Let $I = \int_{-1}^{1} |a_0 + a_1 x + \cdots + a_{n+1} x^{n+1}| \, dx$. Fix $k > 0$ whose value is to be determined. We have two possibilities: either $|a_{n+1}| \leqslant k \max(|a_0|, \ldots, |a_n|)$ or $|a_{n+1}| \geqslant k \max(|a_0|, \ldots, |a_n|)$. In the first case, $I \geqslant \int_{-1}^{1} |a_0 + \cdots + a_n x^n| \, dx - 2|a_{n+1}|/(n+1) \geqslant (K_n^{-1} - 2k/(n+1)) \max(|a_0|, \ldots, |a_n|)$. So, in this case, $\max(|a_0|, \ldots, |a_{n+1}|) \leqslant [\max(k, 1)/(K_n^{-1} - 2k/(n+1))] I$. We have tacitly assumed $K_n^{-1} - 2k/(n+1) > 0$, which will be the case if k is sufficiently small.

In the second case, $\max(|a_0|, \ldots, |a_{n+1}|) \leqslant \max(1/k, 1) |a_{n+1}|$. But by Lemma 10, $|a_{n+1}| \leqslant 2^n I$. So $\max(|a_0|, \ldots, |a_{n+1}|) \leqslant 2^n \max(1/k, 1) I$ in this case.

These two estimates give us $K_{n+1} \leqslant \max(2^n \max(1/k, 1), \max(k, 1)/(K_n^{-1} - 2k/(n+1)))$. Notice that if $a_i = 0$ for $i = 0,..., n-1, n+1$, then $I = 2|a_n|/(n+1)$. Hence, $2K_n/(n+1) \geqslant 1$; and so $K_n^{-1} - 2k/(n+1) > 0$ forces $k < 1$. Hence, our recurrence relation says $K_{n+1} \leqslant \max(2^n/k, 1/(K_n^{-1} - 2k/(n+1)))$. We get the most from this estimate by choosing k so that the two terms in the maximum are equal, i.e., $k/2^n = K_n^{-1} - 2k/(n+1)$. This gives $k = K_n^{-1}/(2^{-n} + 2/(n+1))$. With this choice $K_n^{-1} - 2k/(n+1) > 0$ is guaranteed. Also, this choice gives $K_{n+1} \leqslant 2^n(2^{-n} + 2/(n+1))K_n = (1 + 2^{n+1}/(n+1)) K_n$. ∎

It is not clear whether the estimates given by this proposition are even asymptotically the best possible. A straightforward computation shows that $K_1 = 2$. So at least $K_2 \leqslant 6$, $K_3 \leqslant 22$, etc. Also, since $\|f\|_\infty \leqslant (1 + w_1(f))$, these estimates give $\|f\|_\infty \leqslant (1 + C_n) K_n \|f\|_1$ for all $f \in P_n$. Hence, for $f \in P_n$ and $m(E) \leqslant 1/2(1 + C_n) K_n$ we have $\|1_E f\|_1 \leqslant \|f\|_\infty m(E) \leqslant (1/2)\|f\|_1$. So, if $m(E) \geqslant 2 - 1/2(1 + C_n) K_n$, then E is a determining set for P_n. This is not as good as the result in [5] which says $m(E) \geqslant 2 - 2/(n+3)^2$ is enough.

We now turn to the construction of determining sets for P_n. Suppose that f is continuous and the set $E = \bigcup_{i=1}^n (x_i, x_i + \delta)$, where $\delta > 0$ and $n\delta = 1 + \gamma$ for some γ, $0 < \gamma \leqslant 1$, and $-1 \leqslant x_1 \leqslant x_1 + \delta \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant 1 - \delta$. Then

$$\int_E |f|\, dm = \sum_{i=1}^n \int_{[x_i, x_i+\delta]} |f|\, dm \geqslant \sum_{i=1}^n \delta(|f(x_i)| - w_\delta(f))$$
$$= ((1+\gamma)/2) \left[(2/n) \sum_{i=1}^n |f(x_i)|\right] - (1+\gamma)\, w_\delta(f).$$

Define the discrepancy D_n^* for $(x_1,..., x_n)$ by the formula $D_n^* = \sup_{0 \leqslant \alpha \leqslant 2} |(2/n) \sum_{i=1}^n 1_{[-1,-1+\alpha)}(x_i) - \alpha|$. It is a well-known lemma of numerical integration (see [2]) that $|(2/n) \sum_{i=1}^n |f(x_i)| - \|f\|_1| \leqslant 2w_{D_n^*}(f)$. Also, we can specifically choose $(x_1,..., x_n)$ to minimize the discrepancy if we let $x_i = (2(i-1) - n)/n$, $i = 1,..., n$. The value of D_n^* for this sequence is $2/n$; see [2] again. Hence, putting the estimates together gives $\int_E |f|\, dm \geqslant ((1+\gamma)/2)\|f\|_1 - (1+\gamma)\, w_{D_n^*}(f) - (1+\gamma)\, w_\delta(f)$. Notice $\delta = (1+\gamma)/n \leqslant 2/n$ since $\gamma \leqslant 1$, so $\int_E |f|\, dm \geqslant (1/2)\|f\|_1 + (\gamma/2)\|f\|_1 - 2(1+\gamma)\, w_{2/n}(f)$. With $\gamma > 0$ fixed, as $n \to \infty$, $w_{2/n}(f) \to 0$ and, therefore, for some n large enough, we have $\int_E |f|\, dm \geqslant (1/2)\|f\|_1$.

PROPOSITION 12. *Let $\gamma > 0$ and let E be as above. For E to be a determining set for P_m it suffices to choose the number n defining E to be large enough for $n > V_m/\gamma$, where V_m is the constant $8m(m+1) K_m$.*

Proof. We know that on P_m, $w_{2/n}(f) \leqslant (2/n)(m(m+1)/2) K_m \|f\|_1$. So if we want $w_{2/m}(f) \leqslant (\gamma/4(1+\gamma))\|f\|_1$, we would take $(m(m+1)/n) K_m \leqslant$

$\gamma/4(1+\gamma)$. That is, $n \geqslant 4m(m+1)(1+\gamma)K_m/\gamma$. Since $\gamma \leqslant 1$, $n \geqslant 8m(m+1)K_m/\gamma$ will suffice. ∎

Remarks. The construction actually shows that E is a unique determining set if n is as large as above.

The constant V_m in this proposition is probably not the best one possible, but at least we see that a determining set E_n for P_m need only have $\operatorname{var}(1_E)$ growing larger as $m(E) \to 1$ at a rate proportional to $1/(m(E)-1)$, with the constant of proportion depending only on m. It would be interesting to know if this asymptotic order of growth is best possible. Notice that when $m = 2$, the proposition tells us that we can find a determining set E for the quadratic polynomials with $m(E) = 1 + \gamma$ if we are willing to have the number of intervals comprising E at least $144(1+\gamma)/\gamma$. Better estimates of the constants here might improve this result somewhat. but clearly it is the order $1/\gamma$ which is most important.

One final aspect of this construction is worth observing. Proposition 13 shows that we can construct a sequence sets (E_n) with $m(E_n) > 1$, but $\lim_{n\to\infty} m(E_n) = 1$, such that, for all m, these sets are eventually determining sets for P_m. This suggests that a sequence of determining sets for some P_m may actually be determining for many other subspaces in the limit as $m(E_n)$ approaches 1. In this context, we have the following limitation to this possibility; compare this proposition with Proposition 8.

PROPOSITION 13. *Let (E_n) be a sequence of measurable subsets of $[-1, 1]$ such that $1_{E_n} \to 1/2$ in the w^* topology as $n \to \infty$. Then there is a dense G_δ subset $\mathscr{R} \subset L_1[-1, 1]$ such that for all $f \in \mathscr{R}$, one has $\| 1_{E_n} \|_1 < (1/2)\, \|f\|_1$ for infinitely many n.*

Proof. Define $\mathscr{D}_n = \{f \in L_1[-1, 1] : \| 1_{E_n} f\|_1 \geqslant (1/2)\, \|f\|_1\}$. Then $\mathscr{D}_n$ is a closed set and so is the set $\mathscr{U}_N = \bigcap_{n=N}^{\infty} \mathscr{D}_n$. The set $\bigcup_{N=1}^{\infty} \mathscr{U}_n$ consists of all functions $f \in \mathscr{R}$ such that eventually $\| 1_{E_n} f\|_1 \geqslant (1/2)\, \|f\|_1$. We prove the proposition, with $\mathscr{R} = L_1[-1, 1] \backslash \bigcup_{N=1}^{\infty} \mathscr{U}_N$, by showing that each $\mathscr{U}_N$ has empty interior.

Indeed, suppose that there exists $\varepsilon > 0$ and $f \in L_1[-1, 1]$ such that $F \in \mathscr{U}_N$ whenever $\|F - f\|_1 < \varepsilon$. First, $f \neq 0$ because there exists $F \in L_1[-1, 1]$ with $0 < \|F\|_1 < \varepsilon$ and $\| 1_{E_n} F\|_1 = 0$ as long as $m(E_n) < 1$. That is, no $\mathscr{U}_N$ can have interior around 0. Second, we claim there exists $\delta > 0$ such that $\| 1_{E_n} f\| \geqslant (1/2)\, \|f\|_1 + \delta$ for all $n > N$. If so, this is a contradiction, because $\| 1_{E_n} f\|_1 \to (1/2)\, \|f\|_1$ as $n \to \infty$, and the proof is complete.

We prove this last claim. Since $f \neq 0$, $\gamma = (1/2)\, \|f\|_1 \neq 0$ and $\| 1_{E_n} f\|_1 \geqslant (1/2)\, \|f\|_1 > 0$ for all $n \geqslant N$. Let $\varepsilon_0 = \min(\varepsilon, \gamma)$. We assume without loss of generality that $m(E_n) < 2$ for all $n \geqslant 1$. It is easy to see that there exists a sequence of perturbations $(h_n : n \geqslant N)$ with these properties: (1) $0 < \|h_n\|_1 < \varepsilon_0$ for all $n \geqslant N$, (2) $\|f + h_n\|_1 = \|f\|_1$ for all $n \geqslant N$, and (3)

$\|1_{E_n}(f+h_n)\|_1 < \|1_{E_n}f\|_1 - \varepsilon_0/2$ for all $n \geqslant N$. But then, because $f + h_n \in \mathscr{U}_n$ for all $n \geqslant N$, we have $\|1_{E_n}f\|_1 > (\varepsilon_0/2) + \|1_{E_n}(f+h_n)\|_1 \geqslant (\varepsilon_0/2) + (1/2)\|f+h_n\|_1 = (\varepsilon_0/2) + (1/2)\|f\|_1$. This establishes the claim with $\delta = \varepsilon_0/2$. ∎

References

1. E. W. Cheney and D. E. Wulbert, The existence and unicity of best approximations, *Math. Scand.* **24** (1969), 113–140.
2. L. Kuipers and H. Niederreiter, "Uniform Distribution of Sequences," Wiley, New York, 1974.
3. R. R. Phelps, Cebyšev subspaces of finite dimension in L_1, *Proc. Amer. Math. Soc.* **17** (1966), 646–652.
4. I. Singer, On the extension of continuous linear functional and best approximation in normed linear spaces, *Math. Ann.* **159** (1965), 344–355.
5. D. Amir and Z. Ziegler, Polynomials of extremal L_p-norm on the L_∞-unit sphere, *J. Approx. Theory* **18** (1976), 86–98.